

## COMPUTATIONAL AND INFERENTIAL ORIENTATIONS: LESSONS FROM OBSERVING UNDERGRADUATES READ MATHEMATICAL PROOFS

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*This paper presents selected findings from an assessment of university students' moment-by-moment reading of mathematical proof. This method, adapted from an assessment of narrative reading validated by psychologists, yields novel insights into the strategies students use to construct meaning for the equations in a proof text. In particular, we present evidence that novice readers constructed meanings for the equations using mathematical practices familiar from non-proof oriented courses – substituting and solving for variables – while more experienced readers drew upon practices native to proof-oriented mathematics – inferring properties of quantities. We refer to these as computational and inferential orientations, respectively. We interpret this mismatch of practices in terms of Systemic Functional Linguistic's notion of textual metafunction and briefly discuss its implications for proof-oriented instruction.*

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Researchers of language in mathematics education have long noted that there exists a *mathematical register* (e.g. Halliday, 1975; Pimm, 1987) constituting a subdomain of spoken English (in the English-speaking world). Halliday (1975) explains that the mathematical register constitutes much more than technical terms and the extensive use of symbols. He explains, "It is the meanings, including the styles of meaning and modes of argument, that constitute a register, rather than the words and structures as such" (Halliday, 1975, p. 65). Many have noted that students' ability to participate in mathematics depends on developing competencies within the mathematical register (e.g. Schleppegrell, 2007; Morgan, 1998), but relatively few studies have investigated the nature of those competencies. This paper helps address this gap in the literature.

This report presents findings stemming from a study of undergraduate students' reading of mathematical proofs. We adapted a moment-by-moment think aloud protocol that has proven valuable for assessing narrative text reading competencies (Magliano & Millis, 2003; Magliano, Millis, Team, Levinstein, & Boonthum, 2011). Adapting this methodology to mathematical proof has yielded many insights into novice and experienced student reading behaviors. Hereafter, we highlight one difference between some novice and experienced readers' sense-making about equations in proofs. We refer to their strategies as *computational* and *inferential orientations*.

### Studies of reading behavior

Two mathematics education studies particularly investigated undergraduate students' reading of mathematical text by comparing it to expert reading behaviors. Shepherd and van de Sande (2014) observed that mathematics faculty and graduate students read (i.e. articulated written text) equations in terms of their conceptual structure while undergraduates read them symbol-by-symbol. Inglis and Alcock (2012) used eye-tracking technology to compare undergraduate and mathematician reading of mathematical proofs. They found that experts focused much more on the text surrounding equations, which encoded the logical links in the proof, while undergraduates focused much more on equations themselves. However, eye-tracking methodology did not provide any explanation for why this difference emerged.

As mentioned above, we adapted our methodology from an assessment for narrative reading comprehension developed by Magliano, Millis, and colleagues (Magliano & Millis, 2003; Magliano et al., 2011). That assessment (called RSAT) presents students with narrative text one line at a time and asks students to think aloud or to respond to particular content-related prompts. The validated assessment distinguishes novice and expert readers based on the types of connections they make (not the quality). Students who connect the text to prior information in the text or outside knowledge tend to be more competent readers than those who simply paraphrase or restate the given line. One group of psychologists (Fletcher, Lucas, & Baron, 1999) adapted this method to reading high school proofs. They found that reading such text was more effortful than reading narrative and allowed new types of connections such as anticipating later lines of the text. The findings from moment-by-moment assessment techniques correlate with end-reading comprehension tests, but they reveal a different set of behaviors and insights into the reading process itself. We accordingly adapted the methodology to gain new insights that might not be yielded from existing assessments of proof end-reading comprehension (Mejia-Ramos, Lew, de la Torre, & Weber, 2011) or proof validation (e.g., Alcock & Weber, 2005).

### Conceptualizing the linguistic functions of proof

Most studies of language learning in mathematics education draw upon Halliday's (1975; 1984) theory of Systemic Functional Linguistics (SFL). This theory is particularly useful because it is rooted in language use, it focuses on speakers' and hearers' choices by which they make meaning in language, and it emphasizes the role of linguistic function to explain language use. As Schleppegrell (2004, p. 137) explains, "It is important for students to develop academic register options in different disciplines because particular grammatical choices are functional for construing the kinds of knowledge typical of a discipline." SFL posits that three metafunctions are always operative in the construction or interpretation of a text: the *ideational* metafunction concerns what is being talked about, the *interpersonal* metafunction concerns who is involved in the discourse and how they are positioned, and the *textual* metafunction concerns what type of text is being constructed and what features are necessary or appropriate for such text.

Consider the following excerpt from one of our proof tasks:

For every primitive Pythagorean triple  $(a, b, c)$  there exist some numbers  $s$  and  $t$  with no common factors such that  $s > t \geq 1$  where  $a = st$ ,  $b = \frac{s^2 - t^2}{2}$ , and  $c = \frac{s^2 + t^2}{2}$ .

- 1 Let  $(a, b, c)$  be a primitive Pythagorean triple.
- 2 Then  $a^2 = c^2 - b^2 = (c + b)(c - b)$ .
- 3 We want to show that  $(c + b)$  and  $(c - b)$  are both squares and share no common factors.
- 4 Suppose that  $d$  is a common factor of both  $(c + b)$  and  $(c - b)$ .
- 5 Then  $d$  is also a factor of  $(c + b) + (c - b) = 2c$  and a factor of  $(c + b) - (c - b) = 2b$ .

**Figure 1.** Excerpt from the primitive Pythagorean triples proof (Rotman, 2013).

*Ideationally*, the proof is about characterizing primitive Pythagorean triples. Many portions of the proof are dedicated to deducing numeric properties such as sharing common factors or being perfect squares. *Interpersonally*, mathematics text tends to hide any direct reference to human participants by expressing relations about mathematical objects. In this text, the subjects of most clauses are quantities to which properties are attributed. *Textually*, the symbols and equations mark the text as clearly mathematical. They conform to the genre of mathematical proof by

drawing (and justifying) inferences. The conflict between the claims in lines 3 and 4 further suggest to an expert reader the beginning of a proof by contradiction (or proof that  $d = 1$ ).

A key function that distinguishes mathematical proof from other texts is the emphasis on both making and justifying claims. There is an implicit expectation for the reader to identify the implied warrant in line 5: “the sum and difference of multiples of  $d$  are also multiples of  $d$ .” Students must parse the equation in multiple ways to construct the intended meaning. They must recognize that both sides of the equation represent the same quantity, which is the object of the clause “ $d$  is a factor of.” They must parse the left sides as the sum and difference of the two compound quantities discussed in line 4, which is supported by parenthetical grouping. Finally, a reader should determine that the property of being a multiple of  $d$  is being transferred from two quantities known to share the property to a pair of new quantities  $2b$  and  $2c$ . Though equations are symmetric, this inference is directed (from left-hand quantities to right-hand quantities).

While the various claims about the normative or intended construal of this text are cued in subtle ways in these three lines, an expert reader in many ways constructs its meaning using textual expectations and devices. Ultimately, the text is far from explicit regarding all that it intends to convey. This is not a particular defect of mathematical proof, but rather,

Inferencing on the basis of background assumptions plays a central role in the interpretation of all texts. Highly complex and abstract background assumptions which are not spelled out are often necessary for the interpretation of written language, especially in school contexts. (Schelppegrelle, 2004, p. 11)

With this background we may now better articulate the key question investigated in this report: what textual assumptions and functions do novice readers use to construct meaning for equations in mathematical proofs and how do these differ from those used by more experienced readers?

### Methodology

One contribution of this study is the adaptation of the moment-by-moment think-aloud reading assessment protocol to advanced mathematical proof. Our study involved interviewing mathematics students ranging in experience from no proof-oriented courses to beginning graduate study. To accommodate such diverse experience, we sought out proofs that (a) had at least 10 lines of text, (b) required no knowledge of mathematical concepts with which novices were not familiar, and (c) were relatively outside the curriculum such that experts were less likely to have seen the exact proof being presented. We used four such proofs in our interviews, though due to space limitations we only report on the proof quoted from in Figure 1.

### Adapting moment-by-moment read aloud protocol to proof

Similar to RSAT (Magliano & Millis, 2003), each proof was presented one line at a time. Unlike RSAT, the previous lines of our proofs remained visible. For three of the four proofs students first read relevant definitions and theorems. These remained available for their reference on sheets of paper that doubled as scratch paper. Students were instructed to read each line aloud and think aloud before the interviewer might provide more specific response prompts. We developed the response prompts by listing all the connections that we expect expert readers could make for each line. These included (a) relations to previous lines, (b) implicit warrants, (c) use of definitions, (d) elements of proof frames such as universal generalization, case structure, proof by contradiction, or induction, (e) opportunities to anticipate future lines, and (f) goal statement or achievement. We then decided for which lines we would provide targeted prompts such as “why is this line justified,” “what do you expect in the following lines,” or “why did the author introduce  $d$ .” The interviewers explained the meaning of terms or statements upon request,

though we tried to withhold judgment about the students' interpretations. We note that this methodology does not elicit a naturalistic reading because inviting students to think aloud and respond to targeted prompts in many ways mimics the self-explanation training that Hodds, Alcock, and Inglis (2014) found to improve proof comprehension. Rather, we interpret our data as representing an optimal reading for each student at that time. Such readings still elicit the students' capabilities as readers of mathematical proof.

### Participants and analysis

Participants were recruited from a large public university in the Southwestern United States and a mid-sized public university in the Midwestern United States. Volunteers were solicited from courses ranging from differential equations, introduction to proof, real analysis, and topology (the latter two were dual-listed as undergraduate and graduate courses). To date, 17 students completed interviews for which they were given minor monetary incentives. Three students recruited from differential equations had no collegiate proof coursework and three other participants were recruited from real analysis, which was their first completely proof-oriented course. We classify all six as *novice readers*. The nine other undergraduate students had completed at least one proof-oriented course and were classified as *experienced readers*. Finally, two mathematics graduate students participated who were considered *expert readers*.

Coding began by forming narrative note files for each interview that described students' reading behavior for each line of text. From these line-by-line notes, we formed categories of noteworthy behaviors and patterns in student reading behavior. The two authors independently coded two novice reader interviews in this way and met to discuss and compare the findings. By comparing the various categories of reading behaviors across the four proof tasks, we identified five meta-categories that organized the primary range of categories. We then independently coded the rest of the interviews, meeting regularly to discuss our interpretations and coding. These five meta-categories remained relatively stable, though nuanced to accommodate further cases. The meta-categories are computational versus inferential orientation, depth of encoding for proof claims, logic structure and language, meanings for relevant mathematical concepts, and sense-making activities. Due to space concerns this paper only presents findings about the first category, computational versus inferential orientation.

## Results

The meta-category computational versus inferential orientation most directly addresses students' reading behaviors relevant to the textual metafunction characterized by SFL. In particular, it concerns the mathematical practices that students called upon to make sense of the proof's use of equations. We briefly introduce the two constructs with reference to line 2 in Figure 1. Every study participant could identify and justify the algebraic manipulations the line expressed. This type of reorganization of equations is very common in computational mathematics courses and thus familiar to all of these students. The proof goes on to use this equation in two other ways. First, because  $d$  is defined as a factor of  $(c + b)$  and  $(c - b)$ , this equation supports the inference that  $d$  is a factor of  $a$ . The property "d is a factor" (more precisely  $d^2$  is a factor) is transferred from the right side of the equation to the left. Then, once it is established that  $(c + b)$  and  $(c - b)$  share no common factors, the equation is used to infer that both of those quantities are perfect squares. In this case, the property "is a perfect square" is transferred from the left side of the equation to the two components on the right. This latter use of the equation as a vehicle for inferring properties about the quantities within is not native to early collegiate non-proof-oriented courses. As a result, we noticed much greater discrepancies in students' abilities to construct these meanings for the equations.

We thus define two orientations as follows:

- *Computational orientation*: Mathematics is about computing and solving for numbers, so proofs use equations for counting, evaluating, computing, and solving for quantities.
- *Inferential orientation*: Mathematics is about relating mathematical properties, so proofs use equations for stipulating, inferring, and denying properties of quantities.

We posit a few points about these constructs before presenting some instances of student data that exemplify how these orientations operated in our data. First, the two are not mutually exclusive since both are accurate reflections of mathematical practices. Both of these claims express resources that students may use to interpret linguistic choices in a proof, specifically ideational and textual choices. The main concern is when a proof was constructed according to one orientation and is interpreted according to another. All students recognized that line 2 could be justified in terms of computational practices. However, not all students recognized the uses of line 2 in terms of inferential practices. Also, such orientations are implicit and are merely enacted by students in reading. We now present two cases of novice reading behavior that exemplify reading inferential proof texts using a computational orientation.

#### Nov5 – Explaining operations rather than inferring properties

Nov5 was a chemistry major taking differential equations when he participated in the study. He had no collegiate proof-oriented coursework. He spent about 45 minutes reading the Pythagorean triples proof (17 lines). He expressed low confidence in his interpretations and confusion about what the interviewer meant by questions about “justifying” lines of the proof. His responses to such questions most often focused on explaining the author’s reasoning or on trying to explain why each step in the proof was chosen. He explained line 3 (that introduced subgoals for the proof) saying “I guess they are saying that the focus is going to be on this part of their equation.” When he read line 4 that introduced the quantity  $d$ , he explained:

N5: Ok. I guess they are gonna put in a variable  $d$  that I guess is a factor of both, so it’s both divisible by  $d$  [rising tone as if a question].

I: What is the purpose of introducing  $d$  here?

N5: Uhh. Probably to somehow come out with the quotient for  $b$  and  $c$  being 2.

I: When you say quotient, what were you pointing at?

N5: Just the denominator here [reads  $b$  and  $c$  equations in theorem] just showing where that 2 is coming from. [reads line 5]

I: So why is this true?

N5: Well I am pretty sure what they did was set the parentheses equation up to the  $c^2 - b^2$  and then attempted to solve and then since you would have a square root to solve, you have to have the positive and the negative version of it. I feel like I am wrong, but. [...] Yeah. Multiply it out here. [He multiplies  $(c + b)(c - b)$  on paper.] Well I guess that proves number two for sure. Let’s see what they. I think my idea is off. Ok, now I am not 100% sure where that came from.  $d$  is a common factor.

I: So tell me something about what you are thinking here.

N5: I am thinking maybe possibly that, that these here the  $2b$  and  $2c$  are derivatives. Because the derivative of  $c^2$  would be  $2c$  and the derivative of  $b^2$  would be  $2b$ . I am just trying to think where they got the addition from.

N5: [Later, he reads line 7, which says] “If  $d$  is a factor of both  $b$  and  $c$ , then  $d$  is also a factor of  $a$ .” Ok yeah cause that would match their equation.

I: Which equation are you looking at?

*N5:* Pythagorean theorem because if you mess with one side of the equation you also have to do it with the other side of the equation or else it's not equal, so I guess, yeah.

We notice that Nov5 drew on multiple resources to make meaning from the text. He did not explain the introduction of  $d$  in terms of the goals in line 3, which he only interpreted as focusing the reader's attention. Rather, he hoped to explain why the equations for  $b$  and  $c$  in the theorem both had a denominator of 2. He next cited the algebraic procedure of introducing positive and negative possibilities when taking the root of an equation to explain why the two equations in line 5 differed by addition and subtraction. When he could not recreate the inferred algebraic manipulations, he connected the right sides of the line 5 equations to the derivatives of terms in the line 2 equation. In each case, we notice that Nov5 drew upon computational practices (substituting, solving, and taking derivatives) to make meaning of the text.

Finally, we note that Nov5 was able to make sense of the property inference in line 7, but he explained it in terms of operations performed on each side of the equation rather than in terms of the equality between the quantities. On one hand, we may celebrate his linguistic improvisation for conveying a meaning that he did not seem to have ready tools to express. On the other, his explanations give the sense that he understood " $d$  is a factor of" as an operation to be performed rather than a static property of a number. We expect that an inferential orientation and the language to express its meanings are interdependent and co-emerge (Schleppegrell, 2004).

### **Nov2 – “What is the goal of this equation?”**

Nov2 was an electrical engineering major recruited from differential equations who had not taken any proof-oriented courses. She also spent about 45 minutes reading the Pythagorean triples proof. The following represents her reading of the fourth and fifth lines:

*I:* So what is the purpose of introducing  $d$ ?

*N2:* Umm,  $d$  would,  $d$  would show the correlation between the number  $b$  and  $c$ , to see how they, to figure out what  $c$  and  $b$  is possibly.

*I:* Ok. Anything else that comes to mind about this line?

*N2:* I mean if you multiply everything out and add  $d$  to the equation, you could solve for one of them, if you knew one of the numbers.

*I:* Ok let's see what comes up [shows line 5].

*N2:* [Reads line 5]. Umm. [Long pause.] I'm just seeing how  $d$  fits into everything, I guess?

*I:* My question about this is why is this line justified? Why does this have to be true?

*N2:* Well if you, obviously if you do the math you would figure out that. Sorry I am not good at putting this into words.

*I:* So when you say “do the math” can you tell me more specifically what you are thinking about?

*N2:* Well if. So are we trying to figure out what  $b$  and  $c$  is? What is the goal of this equation?

What are we trying to figure out? Just figuring out what the missing variables are?

N2 showed direct awareness that she was confused about the purpose of introducing  $d$  and the equations in line 5. She inferred that the purposes were to substitute and solve for variables, consistent with the practices in non-proof-oriented courses. She used the parlance “do the math” to refer to algebraic operations, suggesting an identification between the two. When asked to explain, she posed four questions about the purpose and goal of the equations in line 5. These convey a clear sense that her current understanding did not help her construe the text coherently, but she did not know what other goals or practices to infer beyond solving for variables.

By claiming that N2 interpreted using a computational orientation rather than an inferential orientation, we do not claim she did not draw inferences and try to justify them. For instance,

after the proof establishes that  $d$  cannot be a factor of  $b$  and  $c$ , line 10 asserts that “ $d$  is 1 or 2.” N2 responded saying:

*N2: I am guessing it's 2 because, being a factor of 1 is being a factor of everything, right?*

*I: Yeah, 1 is a factor of every number.*

*N2: So since  $d$  is not a factor of  $b$  and  $c$ , it can't be 1. So it would have to be 2.*

It is important to note that the proof's references to the property “share no common factors” up to this point left the caveat “factors greater than 1” unstated. Some other study participants made the same observation that N2 did, but concluded that if 1 is a factor of every number then “share no common factors” must implicitly exclude 1. However, from N2's current construal of the text that  $d$  is not a factor of  $b$  and  $c$ , her inference that  $d = 2$  is justified. As a result, we argue that N2 was clearly capable of drawing (deductive) inferences from the proof, but her inferences did not always match the intended inferences. This further nuances what we mean by students reading with a computational orientation. It describes the resources that students use to make meaning from the text, not their capabilities for engaging in particular mathematical reasoning.

N2 was perturbed enough by the end of reading this proof that she sought to explain herself:

I am really awful at proofs. [...] I am more of a plug and chug person, so I am good at integrating and all that stuff. Word problems like this where I have to think about it and why it equals something, it is not my forte. [...] For me I want to solve, I don't really care about like why is it this, why is it that. I just wanna solve for it. I am just so used to solving, I have never even thought about why is this this and this. It is a little new to me.

Our view of N2's learning capabilities is higher than what she expressed here. We maintain that reading capabilities such as an inferential orientation can be learned through experience with proofs and her difficulty with reading can be overcome. However, her explanation corroborates our claim that her extensive experience with computational practices in mathematics led her to use them to make sense of the proof, even when different practices were intended.

### **Inferential orientation readings of line 5**

The majority of study participants, specifically all of those with more proof experience, constructed meaning for line 5 in terms of attributing the property “ $d$  is a factor of” to two quantities. Furthermore, they interpreted the interviewers' request for justification in a more normative manner rather than simply trying to explain why the author chose to add and subtract the quantities they did. We still observed a range of reading behaviors for justifying the inference. Some students imagined substituting some expression including  $d$  in place of  $(c + b)$  and  $(c - b)$  and noted that  $d$  could be factored out, verifying that the sum and difference should also have  $d$  as a factor. We consider this as essentially *constructing a mini-proof* of the implicit warrant. This type of proof still drew upon algebraic meanings for factor, namely to substitute and “factor out.” Other students – primarily more experienced readers – searched for a general form of the particular inference to evaluate its validity. For instance, they might paraphrase the inference as “if  $d$  is a factor of two numbers, then it is a factor of the sum or difference of those numbers.” This pattern may be called *generalizing the inference*. Finally, one of the graduate student readers justified line 5 by citing that any linear combination of multiples of  $d$  would also be multiples of  $d$ . In this case, she *recalled a warrant* to justify the inference. Thus we recognize that within an inferential orientation students may exhibit a range of meaning-making behaviors.

### **Discussion**

Using observations of students' moment-by-moment reading behavior, we characterize two textual orientations that students drew upon to make meaning for the equations in a proof text: *computational orientations* and *inferential orientations*. We claim this contributes to the existing

literature in two ways. First, our findings could explain Inglis and Alcock's (2012) eye-tracking finding that novice readers focused much more on the equations in a proof rather than the surrounding text. Their calculus student participants may have read the provided proofs using a computational orientation that led them to draw upon the wrong kinds of practices, while the expert readers focused on the inferences expressed by the surrounding text. Second, it begins the process of characterizing some of the reading capabilities that students need to develop to interact productively with advanced mathematical proof texts. Our findings highlight the ways in which students draw upon their knowledge of mathematical practices to read a text. We expect that their ability to understand new practices (such as proving properties of quantities in number theory), to read texts intending to express those practices, and to use language to convey those practices are interdependent and co-emergent in student learning. As noted above, interpreting inferential text in terms of computational mathematical practices was observed only among our least experienced study participants. This could be interpreted in one of two ways. Either basic proof-oriented instruction was successful in fostering an inferential orientation or those who did not construct an inferential orientation did not succeed in proof-oriented courses before we recruited participants. Further data gathering would be required to understand the interaction between these orientations and instruction. We expect that further study should reveal significant variation among the reading behaviors within the inferential orientation. These could be used to understand advanced mathematical reading competencies and how they develop.

## References

- Alcock, L. & Weber, K. (2005). Proof validation in real analysis: Inferring and checking warrants. *Journal of Mathematical Behavior*, 24(2), 125-134.
- Fletcher, C.R., Lucas, S., & Baron, C.M. (1999) Comprehension of Mathematical Proofs. In S.R. Goldman, A.C. Graesser, & P. van den Broek (Eds.) *Narrative Comprehension, Causality, and Coherence: Essays in Honor of Tom Trobasso* (pp. 195-207). Mahwah, NJ: Lawrence Erlbaum Associates.
- Halliday, M. (1978) 'Some aspects of sociolinguistics', in *Interactions Between Linguistics and Mathematics Education*, (pp. 64-73). Copenhagen: UNESCO.
- Hodds, M., Alcock, L., & Inglis, M. (2014). Self-explanation training improves proof comprehension. *Journal for Research in Mathematics Education*, 45(1), 62-101.
- Inglis, M., & Alcock, L. (2012). Expert and novice approaches to reading mathematical proofs. *Journal for Research in Mathematics Education*, 43(4), 358-390.
- Magliano, J. P., & Millis, K. K. (2003). Assessing reading skill with a think-aloud procedure and latent semantic analysis. *Cognition and Instruction*, 21(3), 251-283.
- Magliano, J. P., Millis, K. K., The RSAT Development Team, Levinstein, I., & Boonthum, C. (2011). Assessing comprehension during reading with the Reading Strategy Assessment Tool (RSAT). *Metacognition and learning*, 6(2), 131-154.
- Mejía-Ramos, J. P., Lew, K., de la Torre, J., & Weber, K. (2017). Developing and validating proof comprehension tests in undergraduate mathematics. *Research in Mathematics Education*, 19(2), 130-146.
- Morgan, C. (1998). Writing Mathematically: The Discourse of Investigation. London, UK: Falmer.
- Rotman, J. J. (2013). Journey into mathematics: An introduction to proofs. Courier Corporation.
- Schleppegrell, M. J. (2004). The Language of schooling: A functional linguistics perspective. Mahwah, NJ: Lawrence Erlbaum.
- Schleppegrell, M. J. (2007). The linguistic challenges of mathematics teaching and learning: A research review. *Reading & Writing Quarterly*, 23(2), 139-159.
- Shepherd, M. D., & van de Sande, C. C. (2014). Reading mathematics for understanding—from novice to expert. *The Journal of Mathematical Behavior*, 35, 74-86.